Realization of direct trivial-SPT transition of bosons with U(1)xU(1) symmetry and relation to exactly self-dual easy-plane NCCP1 model

Lesik Motrunich (Caltech)

with Scott Geraedts

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Outline

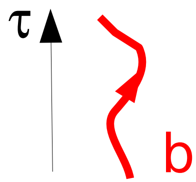
* Review of lattice (2+1)d boson-vortex duality and some successes

* Realization of direct transition between trivial and SPT (integer quantum Hall) phases of bosons with U(1)xU(1) symmetry
  - Hamiltonian formulation, Euclidean action model, phase diagrams and SPT & SET phases
  - direct transition from trivial phase to Integer Quantum Hall phase
  - symmetries of the model
  - special non-local particle-hole-like symmetry
  - relation to exactly-self-dual easy-plane NCCP1 model
  - variations

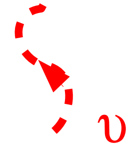
* Conclusions and future directions

Thinking in terms of vortices in (2+1)d

Particle picture:
worldlines of bosons

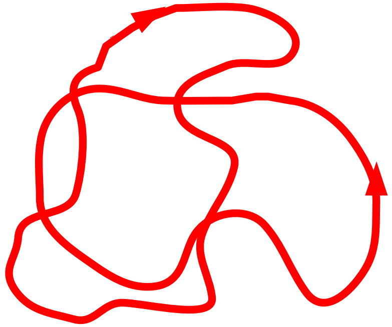


Dual picture in (2+1)d:
worldlines of vortices

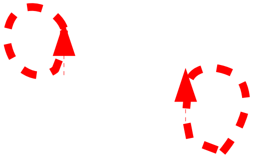


Superfluid phase:

condensed bosons (proliferated worldlines)

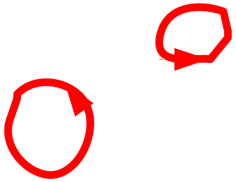


gapped vortices (small worldlines)

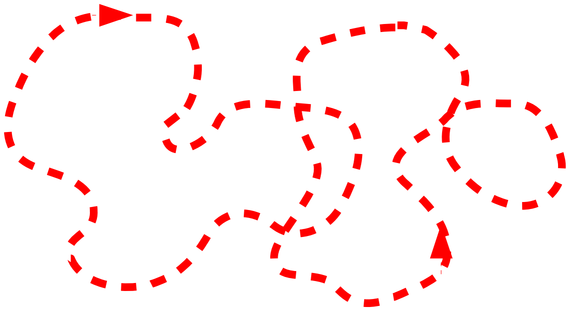


Mott insulator phase:

gapped bosons



proliferated vortices



Successes of thinking in terms of vortices

* Not as successful for describing the phase superfluid-insulator transition in (2+1)d, BUT ...

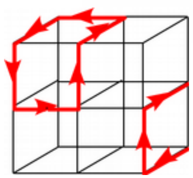
* Very successful for describing non-trivial insulating phases, including with topological order
  - Fractional Quantum Hall systems ("vortex" thinking implicit or explicit; e.g. Chern-Simons flux attachment ~ "vortex attachment")
  - Z_2 fractionalized phase from pair-vortex condensation
  - Insulators with intricate CDW or VBS order from more complex vortex condensations with non-zero wavevectors

* Recent successes in symmetry-protected topological phases:
  - SPT (Integer Quantum Hall effect) phases of bosons
  - SET (fractionalized cousins of SPT) phases of bosons

Precise "duality" transform on a (2+1)D lattice

(Peskin; Halperin & Dasgupta; Fisher & Lee)

$$Z = \sum_{\vec{J},\vec{\nabla}\cdot\vec{J}=0,\vec{J}_{\rm tot}=0} e^{-S[\vec{J}]} = \sum_{\vec{Q},\vec{\nabla}\cdot\vec{Q}=0,\vec{Q}_{\rm tot}=0} \int_{-\infty}^{\infty} D\vec{a}\; e^{-S[\frac{\vec{\nabla}\times\vec{a}}{2\pi}]-i\sum\vec{Q}\cdot\vec{a}}$$



conserved integer-valued 3-currents of bosons on a direct cubic lattice

conserved integer-valued 3-currents of vortices on a dual cubic lattice

"hydrodynamic" real-valued representation of the boson 3-current

$$Z = \sum_{\vec{J},\vec{\nabla}\cdot\vec{J}=0,\vec{J}_{\rm tot}=0} e^{-\frac{1}{2}\sum_k v(k)|\vec{J}(k)|^2} = \sum_{\vec{Q},\vec{\nabla}\cdot\vec{Q}=0,\vec{Q}_{\rm tot}=0} e^{-\frac{1}{2}\sum_k \frac{(2\pi)^2}{v(k)|\vec{f}_k|^2}|\vec{Q}(k)|^2}$$

$$|\vec{f}_k|^2 \equiv \sum_\mu (2 - 2\cos k_\mu) \approx \vec{k}^2$$

Short-range-interacting bosons – "ϕ^4" theory

$$\mathcal{L}_{\rm bos} = |(\vec{\nabla} - i\vec{A}^{\rm ext})\Psi_{\rm bos}|^2 + m_{\rm bos}|\Psi_{\rm bos}|^2 + u_{\rm bos}|\Psi_{\rm bos}|^4 \;\sim\; i\vec{J}_{\rm bos}\cdot\vec{A}^{\rm ext} + \ldots$$

Long-range-interacting vortices – "Higgs model"

$$\mathcal{L}_{\rm vort} = |(\vec{\nabla} - i\vec{a})\Psi_v|^2 + m_v|\Psi_v|^2 + u_v|\Psi_v|^4 + K(\vec{\nabla}\times\vec{a})^2 + i\frac{\vec{\nabla}\times\vec{a}}{2\pi}\cdot\vec{A}^{\rm ext} \;\sim\; i\vec{Q}\cdot\vec{a} + i\frac{\vec{\nabla}\times\vec{a}}{2\pi}\cdot\vec{A}^{\rm ext} + \ldots$$

SPT/SET phases of bosons in (2+1)D in four lines

(Chen et al; Lu & Vishwanath; Senthil & Levin; Geraedts & OIM)

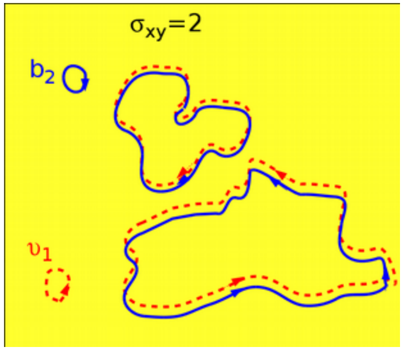
Two species of bosons [U(1)xU(1)]; use dual description for species-1:

$$\mathcal{L}_{\mathrm{vort}-1} = |(\vec{\nabla} - i\vec{a})\Psi_{1,\mathrm{vort}}|^2 + m_{1v}|\Psi_{1,\mathrm{vort}}|^2 + i\frac{\vec{\nabla}\times\vec{a}}{2\pi}\cdot\vec{A}_1^{\mathrm{ext}} + \dots$$

and direct description for species-2:

$$\mathcal{L}_{\mathrm{bos}-2} = |(\vec{\nabla} - i\vec{A}_2^{\mathrm{ext}})\Psi_{2,\mathrm{bos}}|^2 + m_{2,\mathrm{bos}}|\Psi_{2,\mathrm{bos}}|^2 + \dots$$

Consider phase where individual $\Psi_{1,\mathrm{vort}}$, $\Psi_{2,\mathrm{bos}}$ are gapped, while the composite $\Phi \sim (\Psi_{1,\mathrm{vort}})^d (\Psi_{2,\mathrm{bos}})^c$ condensed



$$\mathcal{L}_{\mathrm{composite}} = |[\vec{\nabla} - i(d\vec{a} + c\vec{A}_2^{\mathrm{ext}})]\Phi|^2 + m_\Phi|\Phi|^2 + \dots$$

Φ condensed:

$$\vec{a} \approx -\frac{c}{d}\vec{A}_2^{\mathrm{ext}} \quad \Longrightarrow \quad S_{\mathrm{eff}}[\vec{A}_1^{\mathrm{ext}}, \vec{A}_2^{\mathrm{ext}}] = -i\frac{c}{d}\frac{\vec{\nabla}\times\vec{A}_2^{\mathrm{ext}}}{2\pi}\cdot\vec{A}_1^{\mathrm{ext}}$$

- integer (SPT) Quantum Hall effectof bosons if d=1
- fractional (SET) if d>1

Exact lattice realization of SPT and SET phases of bosons with U(1)xU(1) symmetry

Setup: Two separately conserved species of bosons [U(1)xU(1)]; at integer density ("relativistic"; enforced by unitary particle-hole)

Hamiltonian formulation (Geraedts & OIM 2013):

* Species-1: quantum rotors on "direct" lattice **r**

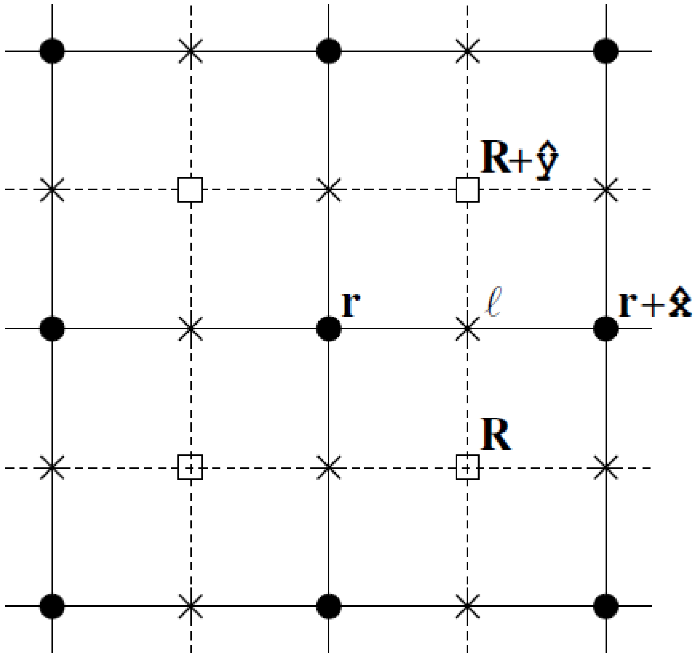
$$[\hat{\phi}_1(\mathbf{r}), \hat{n}_1(\mathbf{r}')] = i\delta_{\mathbf{r},\mathbf{r}'} \ , \quad \phi_1(\mathbf{r}) \in [0, 2\pi] \ , \quad n_1(\mathbf{r}) \in \mathbb{Z}$$

* Species-2: quantum rotors on "dual" lattice **R**

* Harmonic oscillators at intersection points of direct and dual lattice links

$$[\hat{\chi}_\ell, \hat{\pi}_{\ell'}] = i\delta_{\ell,\ell'} \ , \quad \chi_\ell \in \mathbb{R} \ , \quad \pi_\ell \in \mathbb{R}$$

$\hat{\chi}_\ell$ ($\hat{\pi}_\ell$) will affect hopping of the species-1 (species-2) as if they were "gauge fields": Note, however, that they are regular oscillators with finite "gap" and not gauge fields!

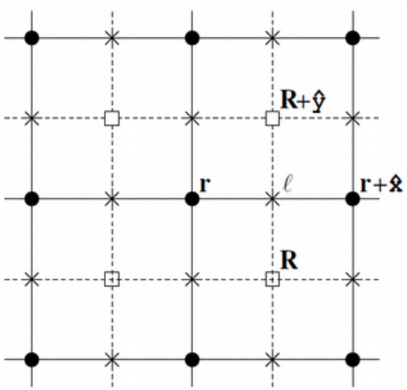


$$\ell = \langle \mathbf{r}, \mathbf{r} + \hat{j} \rangle = \langle \mathbf{R}, \mathbf{R} + \hat{j}' \rangle$$

$$\hat{\alpha}_{1j}(r) = \chi_\ell \ ,$$

$$\hat{\alpha}_{2j'}(R) = \begin{cases} \hat{\pi}_\ell & \text{if } \hat{j}' = \hat{x} \\ -\hat{\pi}_\ell & \text{if } \hat{j}' = \hat{y} \end{cases}$$

Hamiltonian formulation & Euclidean path integral



$$\hat{H} = \hat{H}_{h1} + \hat{H}_{h2} + \hat{H}_{u1} + \hat{H}_{u2} + \hat{H}_{\alpha}$$

$$\hat{H}_{h1} = -\sum_{\mathbf{r},j} h_1 \cos\left[\nabla_j \hat{\phi}_1(\mathbf{r}) - \sqrt{\frac{2\pi}{\eta}}\hat{\alpha}_{1j}(\mathbf{r})\right]$$

← ties "flux" of α_1 to vorticity in ϕ_1

$$\hat{H}_{h2} = -\sum_{\mathbf{R},j} h_2 \cos\left[\nabla_j \hat{\phi}_2(\mathbf{R}) - \sqrt{\frac{2\pi}{\eta}}\hat{\alpha}_{2j}(\mathbf{R})\right]$$

engineer binding of vortices and bosons; composition controlled by η

$$\hat{H}_{u1} = \frac{1}{2}\sum_{\mathbf{r}} u_1 \left[\hat{n}_1(\mathbf{r}) + \sqrt{\frac{\eta}{2\pi}}(\boldsymbol{\nabla} \wedge \hat{\boldsymbol{\alpha}}_2)(\mathbf{r})\right]^2$$

$$\hat{H}_{u2} = \frac{1}{2}\sum_{\mathbf{R}} u_2 \left[\hat{n}_2(\mathbf{R}) + \sqrt{\frac{\eta}{2\pi}}(\boldsymbol{\nabla} \wedge \hat{\boldsymbol{\alpha}}_1)(\mathbf{R})\right]^2$$

← ties "flux" of α_1 to boson number n_2

$$\hat{H}_{\alpha} = \sum_{\ell}\left[\frac{\kappa_1}{2}\hat{\alpha}_1(\ell)^2 + \frac{\kappa_2}{2}\hat{\alpha}_2(\ell)^2\right]$$

← oscillators with finite "gap"

Euclidean space-time path integral (upon integrating out oscillator fields):

$$S = \frac{1}{2}\sum_k v_1(k)\,|\vec{\mathcal{J}}_1(k)|^2 + \frac{1}{2}\sum_k v_2(k)\,|\vec{\mathcal{J}}_2(k)|^2 + i\sum_k w(k)\,[\vec{\nabla}\times\vec{\mathcal{J}}_1](-k)\cdot\vec{\mathcal{J}}_2(k)$$

$$v_{1/2}(k) = \frac{\lambda_{2/1}}{\lambda_1\lambda_2 + \eta^2|\vec{f}_k|^2/(2\pi)^2} + \frac{1}{2h_{1/2}\delta\tau}\,,$$

$$\lambda_\sigma = \delta\tau\kappa_\sigma\eta/(2\pi) = 1/(\delta\tau u_\sigma)$$

$$w(k) = \frac{-\eta}{2\pi}\frac{1}{\lambda_1\lambda_2 + \eta^2|\vec{f}_k|^2/(2\pi)^2}\,.$$

Mixed vortex-boson representation & SPT via condensation of vortex-boson bound state

Dualize $J_1 \rightarrow Q_1$ while leaving J_2 untouched (simple for very large $h_{1,2}$):

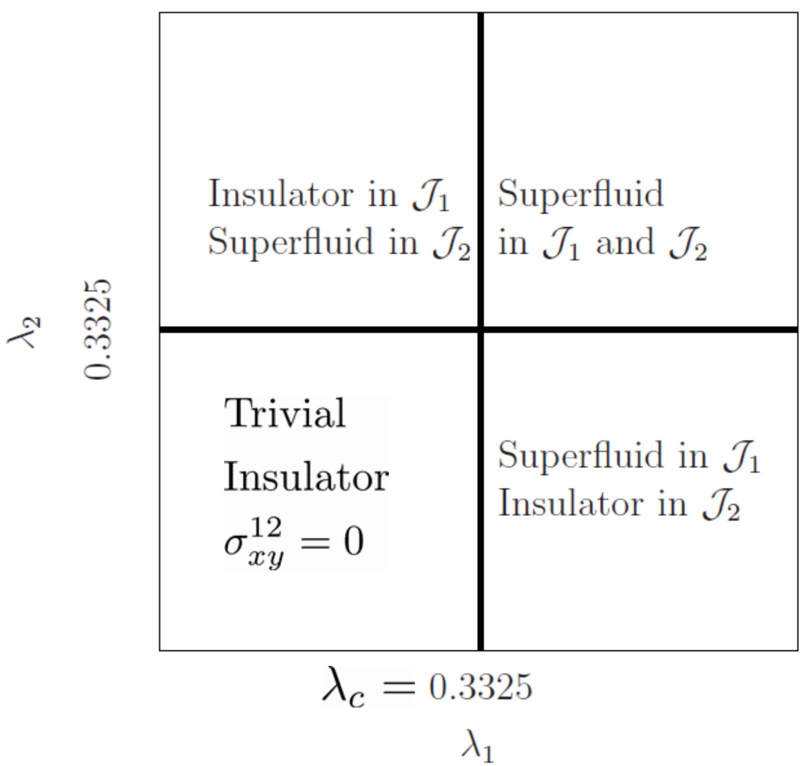
$$S_{QJ} = \sum_k \frac{(2\pi)^2 \lambda_1}{2|\vec{f}_k|^2} |\vec{\mathcal{Q}}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2} \left(\vec{\mathcal{J}}_2(R) - \eta \vec{\mathcal{Q}}_1(R) \right)^2$$

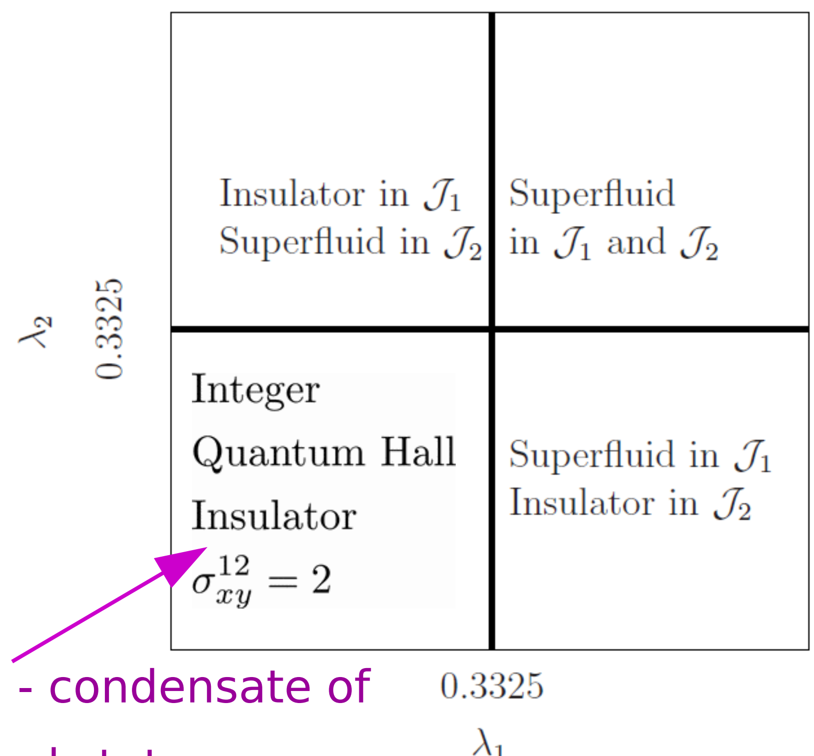
<u>$\eta=0$</u> – two species decouple

$[v_{1/2}(k)=1/\lambda_{1/2}, w=0]$

<u>$\eta=1$</u> – identical phase transition lines to $\eta=0$:

$J_2' = Q_1 - J_2$ and Q_1 decouple



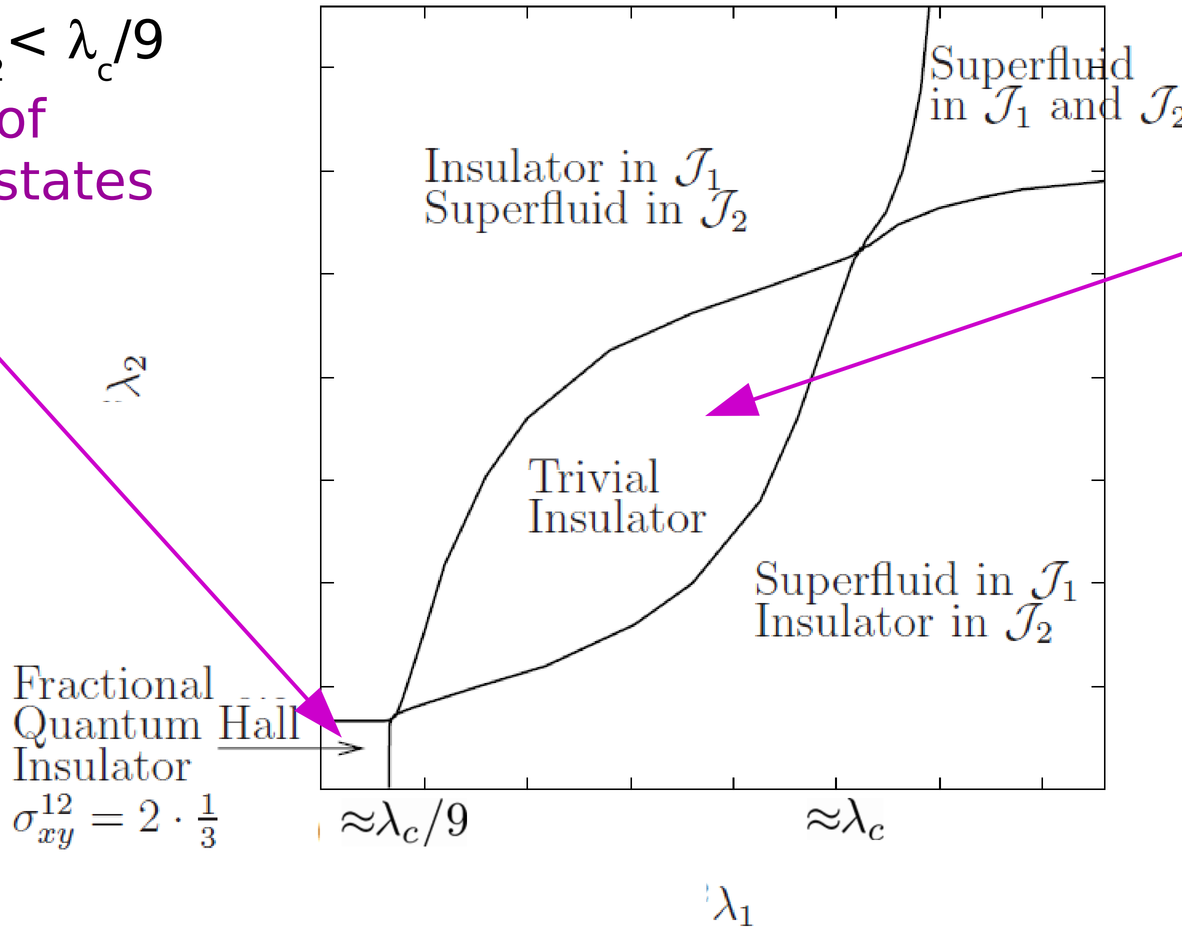


gapped J_2' - condensate of $Q_1=J_2$ bound states

Simple fractional $\eta=1/d$ (e.g. $\eta=1/3$)

$$S_{QJ} = \sum_k \frac{(2\pi)^2\lambda_1}{2|\vec{f}_k|^2}|\vec{\mathcal{Q}}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2}\left(\vec{\mathcal{J}}_2(R) - \frac{1}{3}\vec{\mathcal{Q}}_1(R)\right)^2$$

Very small $\lambda_{1/2} < \lambda_c/9$
- condensate of $Q_1=3J_2$ bound states

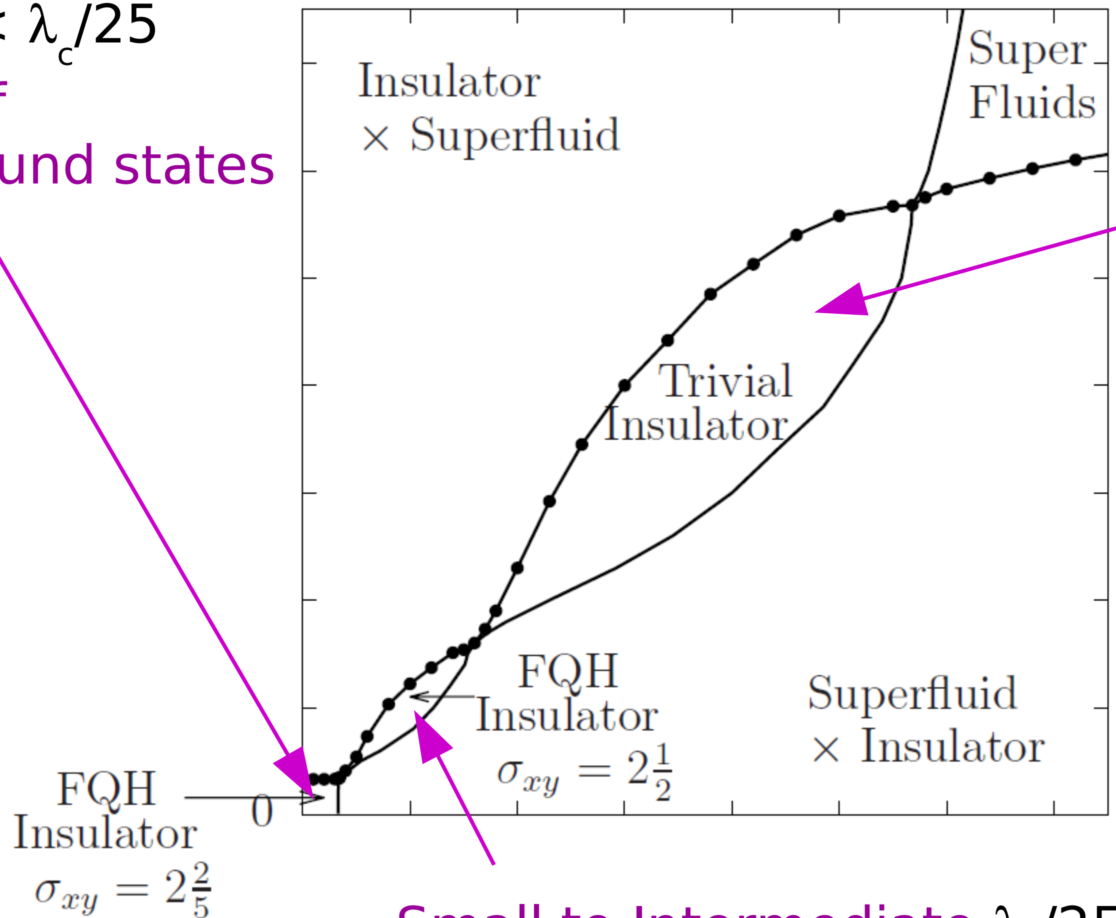


Intermediate
$\lambda_c/9 < \lambda_{1/2} < \lambda_c$
- condensate of individual Q_1
(& gapped J_2)

More complex rational example: $\eta=2/5$ - hierarchy of fractionalized phases

$$S_{QJ} = \sum_k \frac{(2\pi)^2\lambda_1}{2|\vec{f}_k|^2}|\vec{\mathcal{Q}}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2}\left(\vec{\mathcal{J}}_2(R) - \frac{2}{5}\vec{\mathcal{Q}}_1(R)\right)^2$$

Very small $\lambda_{1/2} < \lambda_c/25$
- condensate of $(Q_1,J_2)=(5,2)$ bound states

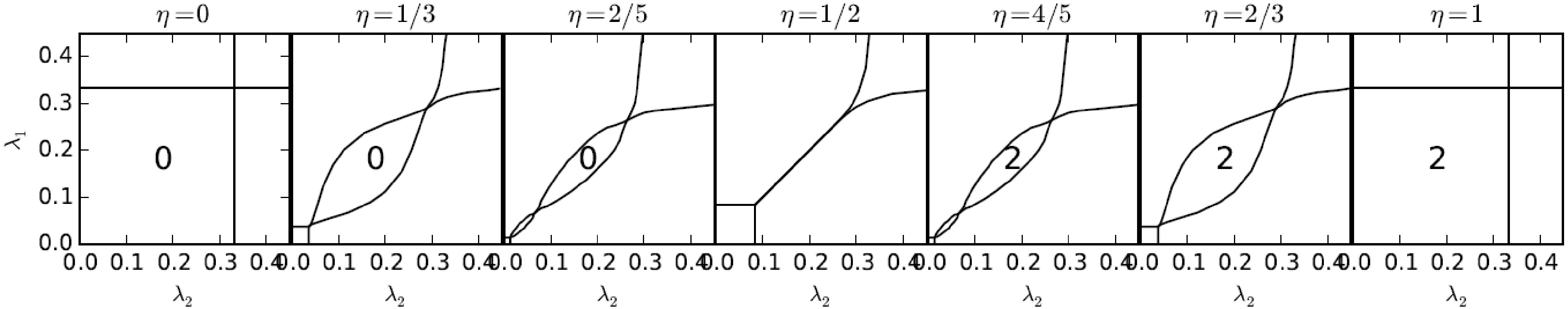


Intermediate $\lambda_c/4 < \lambda_{1/2} < \lambda_c$
- condensate of individual Q_1 (& gapped J_2)

Small to Intermediate $\lambda_c/25 < \lambda_{1/2} < \lambda_c/4$
- condensate of $(Q_1,J_2)=(2,1)$ bound states

Phase diagram: cuts at fixed η

$$S_{QJ} = \sum_k \frac{(2\pi)^2 \lambda_1}{2|\vec{f}_k|^2} |\vec{\mathcal{Q}}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2} \left(\vec{\mathcal{J}}_2(R) - \eta \vec{\mathcal{Q}}_1(R) \right)^2$$



* For all η, the trivial insulator persists in the window $\lambda_c/4<\lambda_{1/2}<\lambda_c$ (topo phase requires at least Q_1=2), but is compressed towards $\lambda_1=\lambda_2$ line

* η = 1/2 – similar to η =1/d (e.g., there is topo phase at small $\lambda_{1/2}$) except that in our model the trivial phase collapses to a line, which we found to be 1st-order transition (Geraedts and OIM 2011)

* η >1/2 - phase diagrams can be obtained from η < 1/2 using change of vars. $J_2' = Q_1 - J_2$. Under this $J_2 - \eta Q_1 = (1-\eta) Q_1 - J_2'$ – relates models at η and η'=1-η. This relates phases with physical Hall conductivities σ_{xy} and 2-σ_{xy}

Careful look at symmetries: what allowed exact knowledge of the transition?

$$S_{JJ} = \frac{1}{2}\sum_k v_1(k)\ |\vec{\mathcal{J}}_1(k)|^2 + \frac{1}{2}\sum_k v_2(k)\ |\vec{\mathcal{J}}_2(k)|^2 + i\sum_k w(k)\ [\vec{\nabla}\times\vec{\mathcal{J}}_1](-k)\cdot\vec{\mathcal{J}}_2(k)$$

* Unitary particle-hole C: $J_s \to -J_s$ (fixes to "integer density")
* Anti-unitary T_{-+}: $J_1 \to -J_1$, $J_2 \to J_2$, i->-i (NOT usual time reversal; allows non-zero σ_{xy}^{12}; allows Monte Carlo simulations)
* Species interchange symm. if $v_1=v_2$ (smth we can require-need for direct transition)

<u>Mixed vortex-boson vars:</u> $$S_{QJ} = \sum_k \frac{(2\pi)^2\lambda_1}{2|\vec{f}_k|^2}|\vec{\mathcal{Q}}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2}\left(\vec{\mathcal{J}}_2(R) - \eta\vec{\mathcal{Q}}_1(R)\right)^2$$

* Property of "invariance" under $\eta \to 1-\eta$, $J_2 \to Q_1 - J_2$ – NOT a symmetry as it relates models with different η (reminiscent of particle-hole transformation of electrons in the lowest Landau level).
  - At $\eta = 1/2$, it can be viewed as a symmetry of the model and is responsible for putting the model exactly at the transition!
  - This transformation is particle-hole-like ("-" in front of J_2), is simple to state in S_{QJ} but (nearly) impossible in the boson vars S_{JJ} – "non-local"? Expect that it is anti-unitary since Q_1 is unchanged.

Non-local particle-hole-like symmetry in the Hamiltonian formulation?

The current loop model that gave "nice" S_{QJ} with nice "particle-hole-like" transformation was obtained in the limit of very large (infinite) $h_{1,2}$

$$\hat{H}_{h1} = -\sum_{\mathbf{r},j} h_1 \cos\left[\nabla_j \hat{\phi}_1(\mathbf{r}) - \sqrt{4\pi}\hat{\alpha}_{1j}(\mathbf{r})\right]$$

$$\hat{H}_{h2} = -\sum_{\mathbf{R},j} h_2 \cos\left[\nabla_j \hat{\phi}_2(\mathbf{R}) - \sqrt{4\pi}\hat{\alpha}_{2j}(\mathbf{R})\right]$$

← specialized to $\eta=1/2$; the two terms commute

Very large (infinite) $h_{1,2}$ → define restricted Hilbert space by

$$\exp\left(i[\nabla_j \hat{\phi}_1(\mathbf{r}) - \sqrt{4\pi}\hat{\alpha}_{1j}(\mathbf{r})]\right) = 1 \ , \quad \exp\left(i[\nabla_j \hat{\phi}_2(\mathbf{R}) - \sqrt{4\pi}\hat{\alpha}_{2j}(\mathbf{R})]\right) = 1 \ .$$

In this restricted Hilbert space

$$\sqrt{4\pi}(\boldsymbol{\nabla} \wedge \hat{\boldsymbol{\alpha}}_1)(\mathbf{R}) = -2\pi Q_1(\mathbf{R}) \ , \quad \sqrt{4\pi}(\boldsymbol{\nabla} \wedge \hat{\boldsymbol{\alpha}}_2)(\mathbf{r}) = -2\pi Q_2(\mathbf{r}) \ , \quad Q_1, Q_2 \in \mathbb{Z}$$

The u-terms have the form

$$\hat{H}_{u1} = u_1 \left[\hat{n}_1(\mathbf{r}) + \frac{1}{\sqrt{4\pi}}(\boldsymbol{\nabla} \wedge \hat{\boldsymbol{\alpha}}_2)(\mathbf{r})\right]^2 = u_1 \left[\hat{n}_1(\mathbf{r}) - \frac{1}{2}Q_2(\mathbf{r})\right]^2 \ , \quad \hat{H}_{u2} = u_2 \left[\hat{n}_2(\mathbf{R}) - \frac{1}{2}Q_1(\mathbf{R})\right]^2$$

– invariant under $n_1 \to Q_2 - n_1 \ , \quad n_2 \to Q_1 - n_2$

Resembles PH symmetry of electrons in the LLL (very large $h_{1,2}$ is "quenching" the boson kinetic energy) but works only at $\eta=1/2$

Relation of η=1/2 to exactly-self-dual "easy-plane NCCP1"

$$S_{QJ} = \sum_k \frac{(2\pi)^2\lambda_1}{2|\vec{f}_k|^2}|\vec{\mathcal{Q}}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2}\left(\vec{\mathcal{J}}_2(R) - \frac{1}{2}\vec{\mathcal{Q}}_1(R)\right)^2$$

Change of vars in the partition sum: $\vec{\mathcal{Q}}_1 = \vec{\mathcal{L}}_1 + \vec{\mathcal{L}}_2 \ , \quad \vec{\mathcal{J}}_2 = \vec{\mathcal{L}}_2$

$$S_{LL}[\vec{\mathcal{L}}_1, \vec{\mathcal{L}}_2] = S_{QJ}[\vec{\mathcal{L}}_1 + \vec{\mathcal{L}}_2, \vec{\mathcal{L}}_2] \ = \sum_k \frac{(2\pi)^2\lambda_1}{2|\vec{f}_k|^2}|\vec{\mathcal{L}}_1(k) + \vec{\mathcal{L}}_2(k)|^2 + \sum_R \frac{1}{8\lambda_2}(\vec{\mathcal{L}}_2(R) - \vec{\mathcal{L}}_1(R))^2$$
$$= \frac{1}{2}\sum_k \left[v_+(k)|\vec{\mathcal{L}}_1(k) + \vec{\mathcal{L}}_2(k)|^2 + v_-(k)|\vec{\mathcal{L}}_2(k) - \vec{\mathcal{L}}_1(k)|^2\right]$$

Long-range $v_+(k) \sim 1/k^2$ & short-range $v_-(k)$ – the structure is qualitatively the same as EP-NCCP1 with (Euclidean) Lagrangian (OIM & Vishwanath; Senthil et al)

$$i(\vec{\mathcal{L}}_1 + \vec{\mathcal{L}}_2)\cdot\vec{a} + K(\vec{\nabla}\times\vec{a})^2 + \text{s.r.int.}$$

* Original boson unitary particle-hole C: $L_s \rightarrow -L_s$
* Original boson anti-unitary T_{-+}: $L_s \rightarrow L_s$, i->-i
* Original η =1/2 produced S_{LL} symmetric under $L_1 \leftrightarrow L_2$ interchange;

invariance of S_{QJ} under $J_2 \rightarrow Q_1 - J_2$, $Q_1 \rightarrow Q_1$ equivalent to symm. of S_{LL} under $L_1 \leftrightarrow L_2$

Relation to exactly-self-dual "easy-plane NCCP1"

$$S_{LL}[\vec{\mathcal{L}}_1, \vec{\mathcal{L}}_2] = \frac{1}{2}\sum_k \left[v_+(k)|\vec{\mathcal{L}}_1(k) + \vec{\mathcal{L}}_2(k)|^2 + v_-(k)|\vec{\mathcal{L}}_2(k) - \vec{\mathcal{L}}_1(k)|^2 \right]$$

Duality transformation from L_1, L_2 to M_1, M_2 gives

$$S_{MM}[\vec{\mathcal{M}}_1, \vec{\mathcal{M}}_2] = \frac{1}{2}\sum_k \left[\frac{\pi^2}{v_+(k)|\vec{f}_k|^2}|\vec{\mathcal{M}}_1(k) + \vec{\mathcal{M}}_2(k)|^2 + \frac{\pi^2}{v_-(k)|\vec{f}_k|^2}|\vec{\mathcal{M}}_2(k) - \vec{\mathcal{M}}_1(k)|^2 \right]$$

short-range if $v_+(k)$ is long-range — long-range if $v_-(k)$ is short-range

S_{MM} theory has qualitatively similar structure as S_{LL} up to change of sign of one of the currents (OIM & Vishwanath). "**Exact self-duality**" in the sense

$$S_{MM}[M_1, M_2] = S_{LL}[M_2, -M_1]$$

can be achieved if

$$v_+(k)\, v_-(k) = \pi^2/|k|^2$$

* Original boson interchange symmetry $\lambda_1=\lambda_2$ → exact EP-NCCP1 self-duality.

Our model at $\eta=1/2$ & $\lambda_1=\lambda_2$ <-> exactly-self-dual EP-NCCP1!

Future directions

* The original model that was the simplest to simulate showed first-order transition. However, whole class of models with $v_+(k)\, v_-(k) = \pi^2/|k|^2$ are also exactly at the transition – harder but worthwhile to study in Monte Carlo.
 - Recent QMC studies of bosonic SPT-trivial transition and easy-plane VBS-superfluid transition find 2nd-order transitions(Slagle, You, & Xu; Qi et al)
 - EP NCCP1 model is also related to exactly-self-dual fermionic N_f=2 QED3 (Wang et al, Benini et al, Mross et al)

* Models with direct transition but without non-local particle-hole symm. – "emergent interchange symmetry" between L_1 and L_2 in the NCCP1? ~ "emergent non-local particle-hole symm" at the trivial-BIQHE transition?

* (Non-relativistic) Bosons at finite density in external field - can realize SPT & SET phases in such models as well; strong constraints on allowed σ_{xy}. He et al: direct transition between σ_{xy}=2 and σ_{xy}=-2 & proposal of $(\text{NCCP1})^2$ criticality. So far, our models go through intermediate phases.

* Lattice duality in the NCCP1 model without assuming easy-plane?

Variation: are local symmetries enough for direct transition?

(Probably) YES – motivated by observation of direct bosonic trivial-SPT transition in the Hubbard on bilayer honeycomb (Slagle, You, & Xu; Qi et al. - different model with many symmetries)

In our models, if $v_1=v_2$, then by the boson interchange symmetry we cannot pass through phase (J_1 superfluid, J_2 insulating) or (J_1 insulating, J_2 superfluid) – except their meeting line (coexistence or 2nd-order). It is then natural to expect direct transition from trivial to BIQH. Without the non-local symmetry we do not know the exact critical point, but can search numerically.

Question: What does this correspond to in the variables that gave us EP-NCCP1 model?

$$S_{LL}[\vec{\mathcal{L}}_1, \vec{\mathcal{L}}_2] = \frac{1}{2} \sum_k \left[\frac{(2\pi)^2}{v_1(k)|\vec{f}_k|^2} \left| \vec{\mathcal{L}}_1(k) + \left(1 + \frac{w(k)|\vec{f}_k|^2}{2\pi}\right) \vec{\mathcal{L}}_2(k) \right|^2 + v_2(k)|\vec{\mathcal{L}}_2(k)|^2 \right] ;$$

$$S_{MM}[\vec{\mathcal{M}}_1, \vec{\mathcal{M}}_2] = \frac{1}{2} \sum_k \left[v_1(k) \left| \vec{\mathcal{M}}_1(k) \right|^2 + \frac{(2\pi)^2}{v_2(k)|\vec{f}_k|^2} \left| \vec{\mathcal{M}}_2(k) - \left(1 + \frac{w(k)|\vec{f}_k|^2}{2\pi}\right) \vec{\mathcal{M}}_1(k) \right|^2 \right]$$

Variation: are local symmetries enough for direct transition?

Original species interchange symmetry if $v_1 = v_2$ → exact self-duality in the sense

$$S_{MM}[M_1, M_2] = S_{LL}[M_2, -M_1]$$

Without the non-local particle-hole symmetry, the S_{LL} is not symmetric under $L_1 \leftrightarrow L_2$ interchange – **NOT the original EP NCCP1 model.**

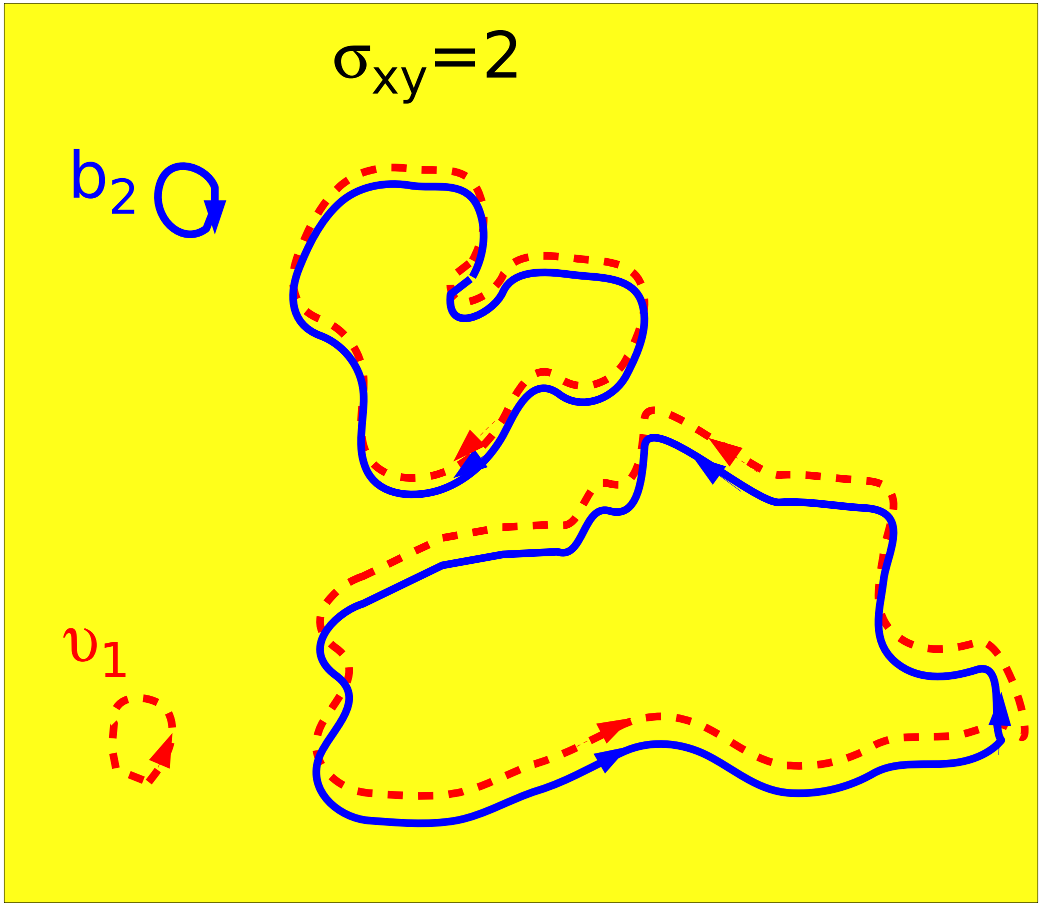
Nevertheless, the above self-duality implies that energetics of M_1 is identical to energetics of L_2 and hence guarantees that either:

1) M_1 & L_2 are simulatenously gapped (L_1 condensed, L_2 gapped – trivial insulator)
2) M_1 & L_2 are simulatenously condensed (L_1 gapped, L_2condensed – SPT insulator)
3) M_1 & L_2 are simulatenously at transition – critical or 1^{st}-order (direct trivial – SPT transition)

Just self-duality does NOT correspond to criticality but guarantees that once the transition is found, both L_1 and L_2 are simultaneously “critical” - is there an “emergent interchange symmetry” between L_1 and L_2?

SPT/SET phases of bosons in (2+1)D in four lines

(Chen et al; Lu & Vishwanath; Senthil & Levin; Geraedts & OIM)



“Duality” is constrained analog of

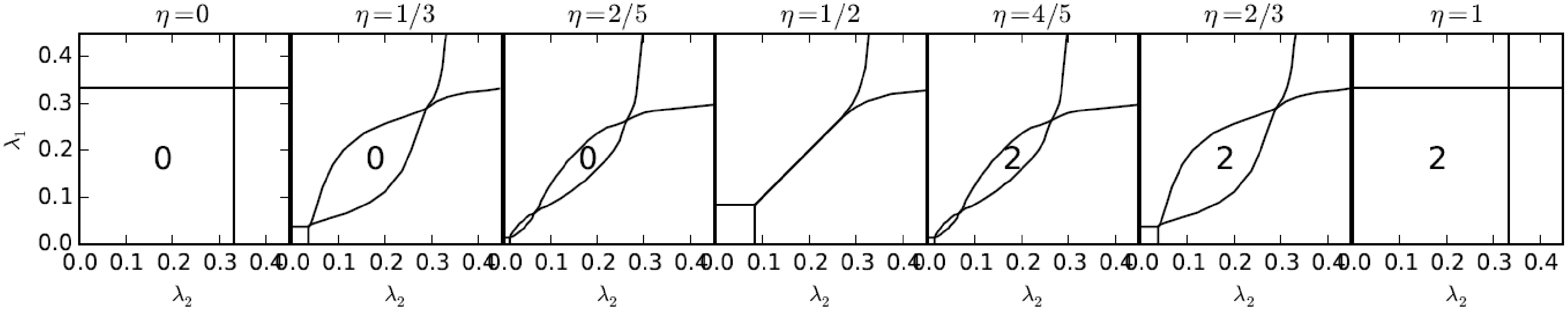
$$\sum_{X=-\infty}^{\infty} = \int_{-\infty}^{\infty} dx \sum_{p=-\infty}^{\infty} e^{-i2\pi px}$$

Can argue using precise relation to Villain model (Peskin; Halperin & Dasgupta) that Q's can be thought of as vortices

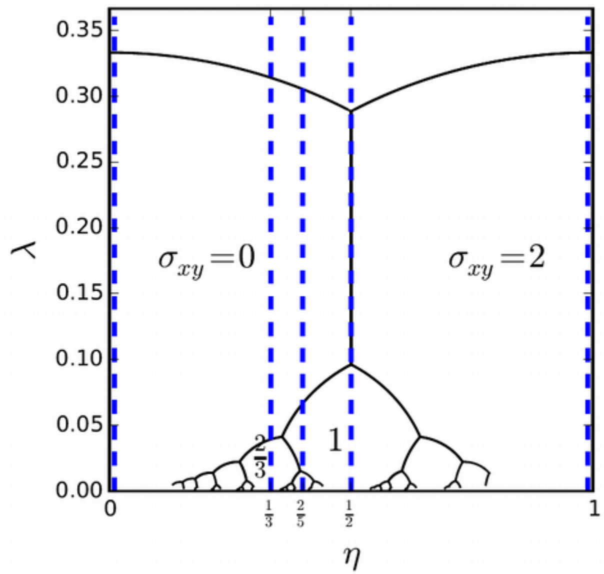
Phase diagram

$$S_{QJ} = \sum_k \frac{(2\pi)^2\lambda_1}{2|\vec{f}_k|^2}|\vec{\mathcal{Q}}_1(k)|^2 + \sum_R \frac{|\vec{\mathcal{J}}_2(R) - \eta\vec{\mathcal{Q}}_1(R)|^2}{2\lambda_2}$$

Cuts at fixed η:



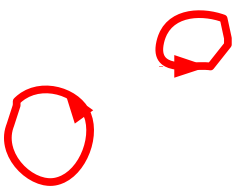
Cut along the λ_1=λ_2 line:



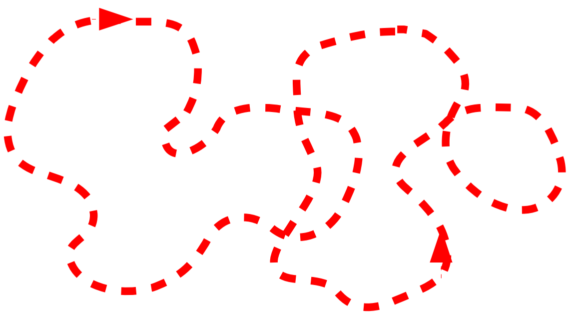
Conventional Mott insulator via vortices

Mott insulator <---> condensate of vortices:

gapped bosons



proliferated vortices



Excitations in the Mott insulator:

No gapless modes <---> "Higgs mechanism”

Original boson <---> vortex in the vortex field Ψ_v. N.B.: Abrikosov-Nielsen vortices in the Higgs model have short-ranged interactions

Charge quantization <---> flux quantization for A-N vortices in Ψ_v
Charge 1 <---> 2π flux of a == "unit flux" "h_{vort}""c_{vort}"/"q_{vort}”

Z_2 fractionalized Mott insulator via vortices

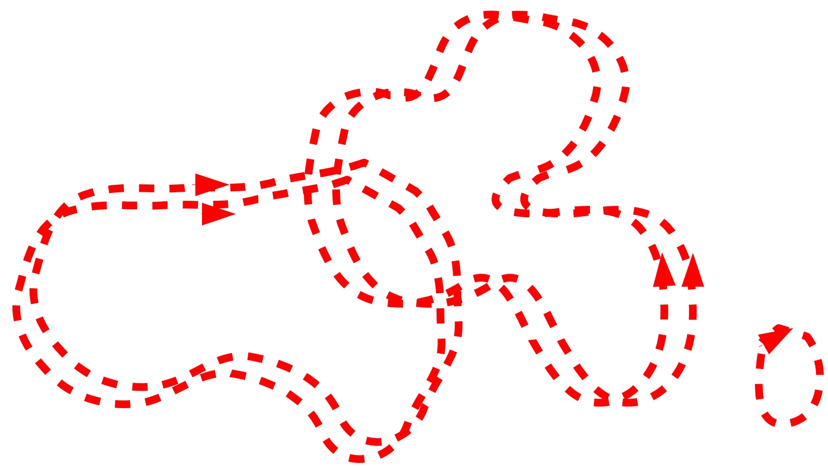
(Balents, Fisher, and Nayak; Senthil and Fisher)

Usefulness of dual language: simple states in terms of vortices can be non-trivial states in terms of original bosons!

Z_2 fractionalized phase:

?

condensed pairs of vortices



* Featureless Mott insulator (no gapless modes, no order)
* Charged excitations <---> vortices in $\Psi_{\text{pair-vort}} \sim (\Psi_v)^2$
* Charge quantum <---> new flux quant. "h_{vort}""c_{vort}"/(2"q_{vort}") = 1/2!
* Gapped "vison" <---> unpaired vortex
* Chargon and vison have mutual π statistics

Vortex thinking in Fractional Quantum Hall

Fermions at ν=1/m, with m=odd.
Flux attachment:

$$\Psi_{\text{ferm}} = \left(\prod_{i<j} \frac{z_i - z_j}{|z_i - z_j|} \right)^m \Psi_{\text{bos}}$$

---> Chern-Simons field theory;
- feels like we are attaching m vortices and condensing vortex-charge composites;

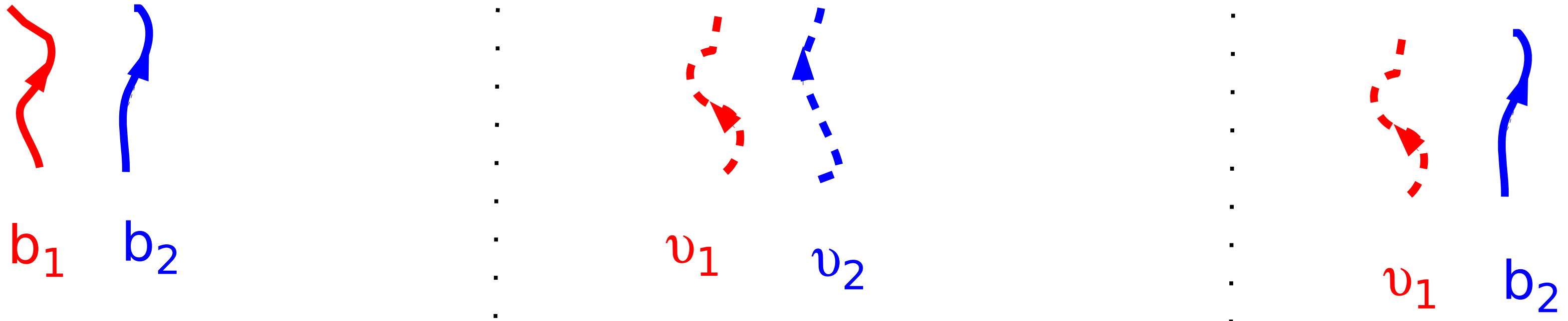
- vortex language is used often, e.g., when discussing excitations; in Lee-Fisher hierarchy construction; and Wen's K-matrix formulation

N.B.: Difficult to put C-S theory on a lattice/make precise (~cannot simultaneously work with discrete particles and discrete vortices). Our U(1)xU(1) systems allow solution without using flux attachment and avoid any such difficulties – everything can be made precise.

Bosons with U(1)xU(1) symmetry

- two species of separately conserved bosons

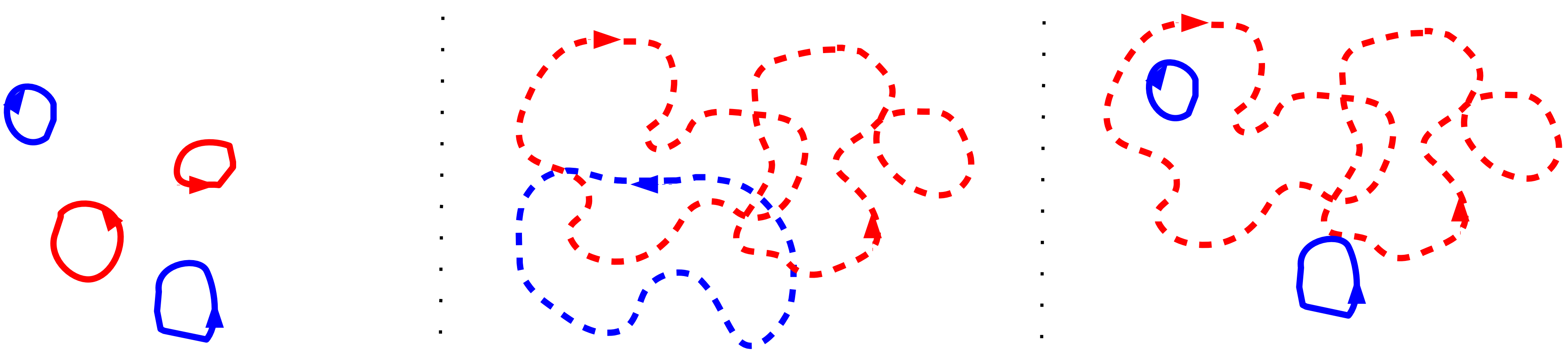
Particle picture: Vortex picture: Mixed picture:



Conventional Mott insulator

gapped bosons

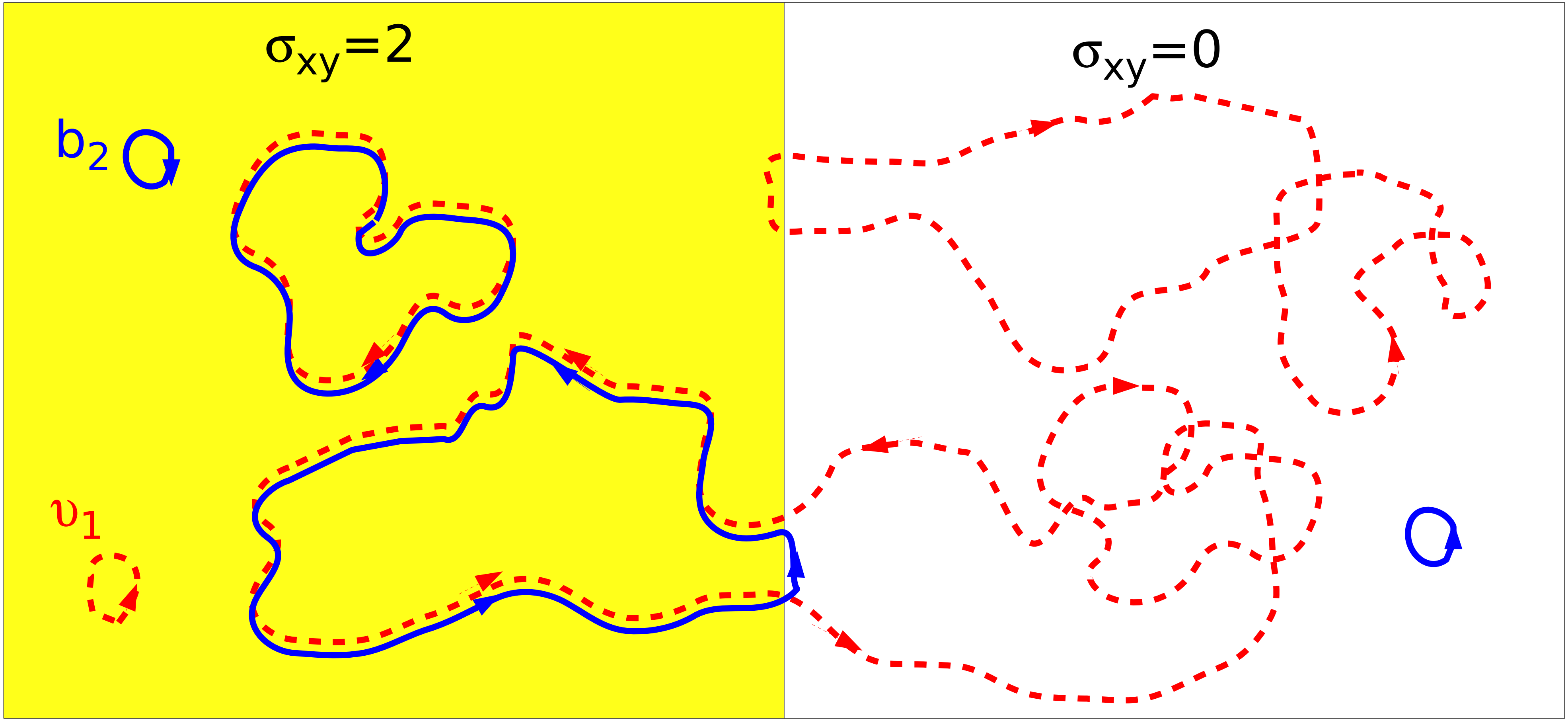
proliferated vortices



Integer quantum Hall insulator $\sigma^{12}_{xy}=2$

(Chen et. al.; Lu and Vishwanath; Senthil and Levin)

Mixed picture using υ_1 and b_2 is particularly convenient: condensation of bound states of υ_1 and b_2

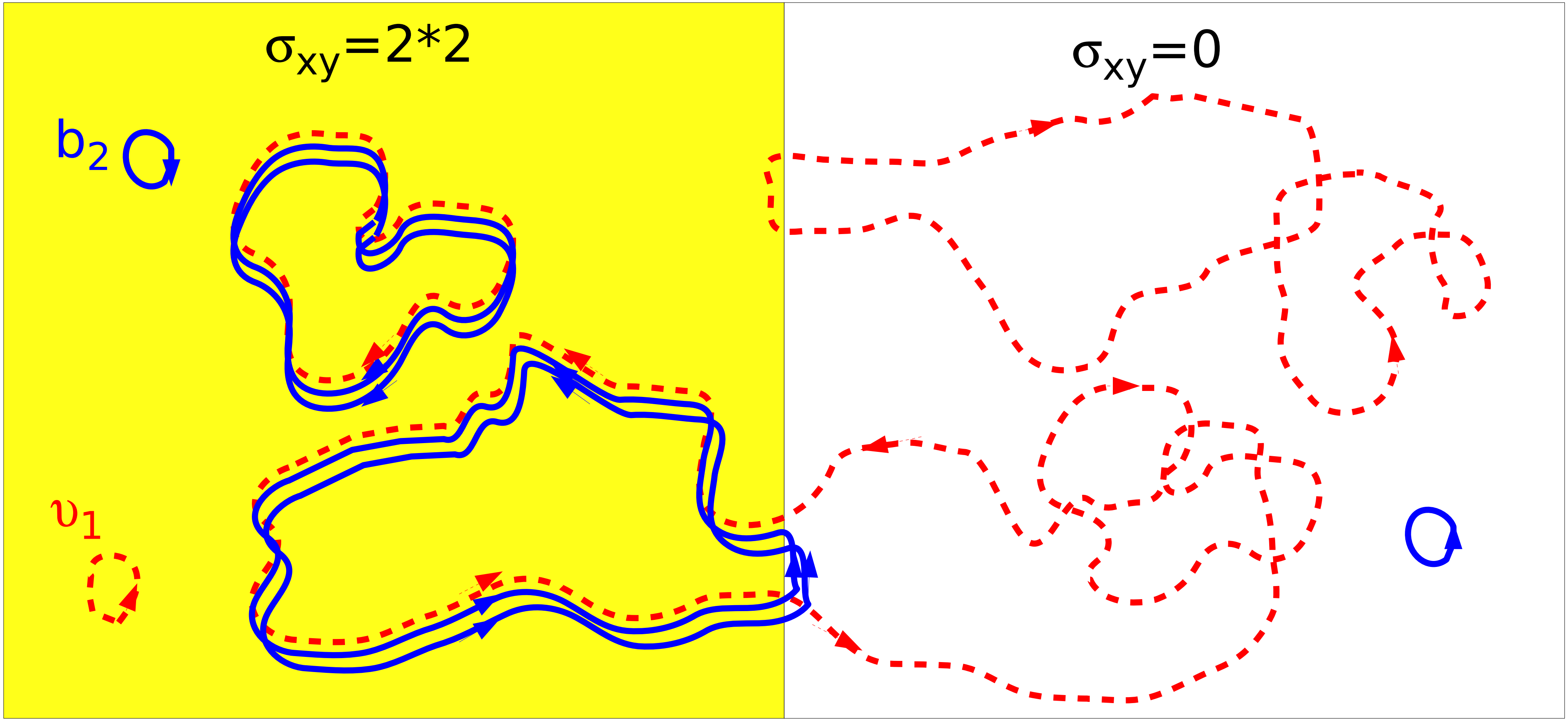


mismatch at the boundary -> edge states!

Integer quantum Hall insulator $\sigma^{12}_{xy}=2n$

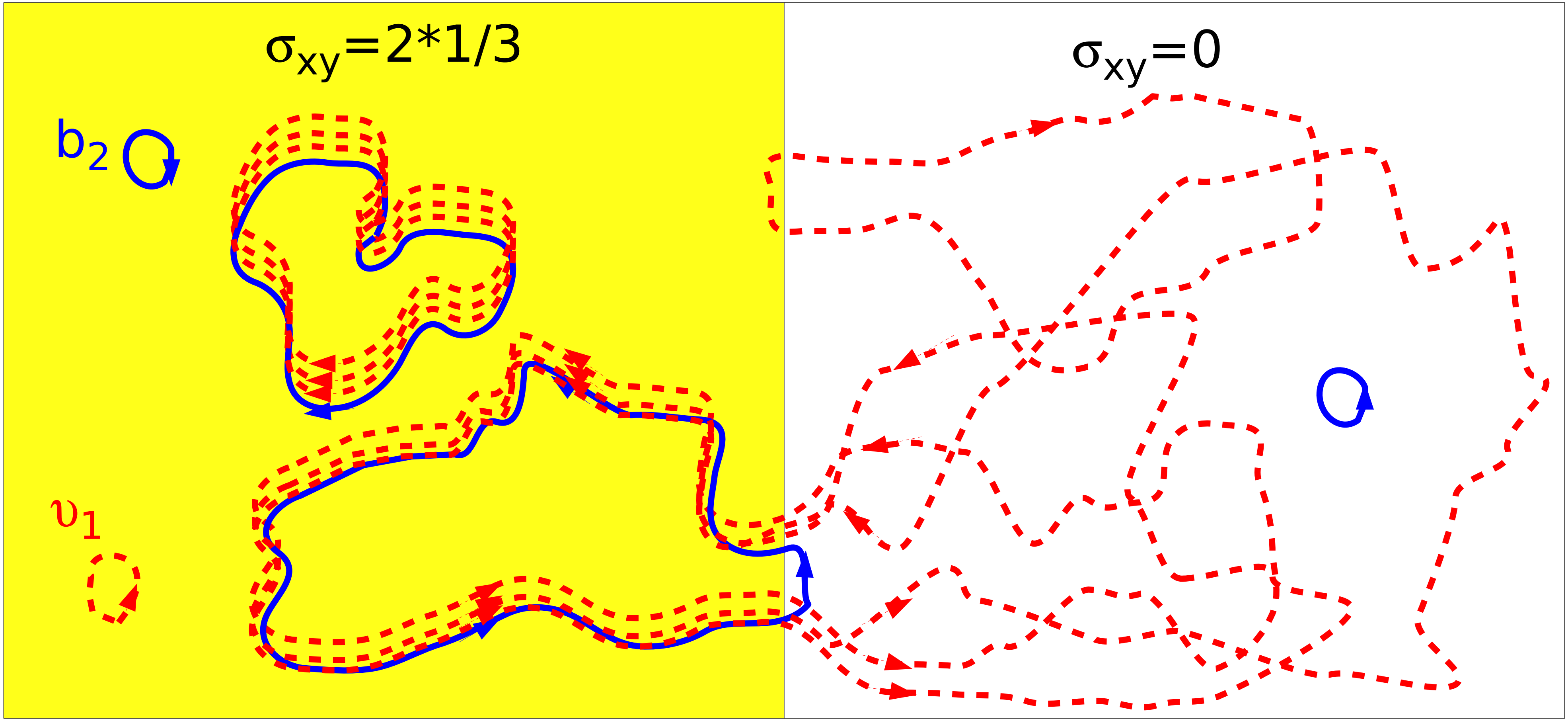
(Chen et. al.; Lu and Vishwanath; Senthil and Levin)

condensation of bound states of υ_1 and $(b_2)^n$



Fractional quantum Hall insulator $\sigma^{12}{}_{xy}=2c/d$

condensation of bound states of $(\upsilon_1)^d$ and $(b_2)^c$



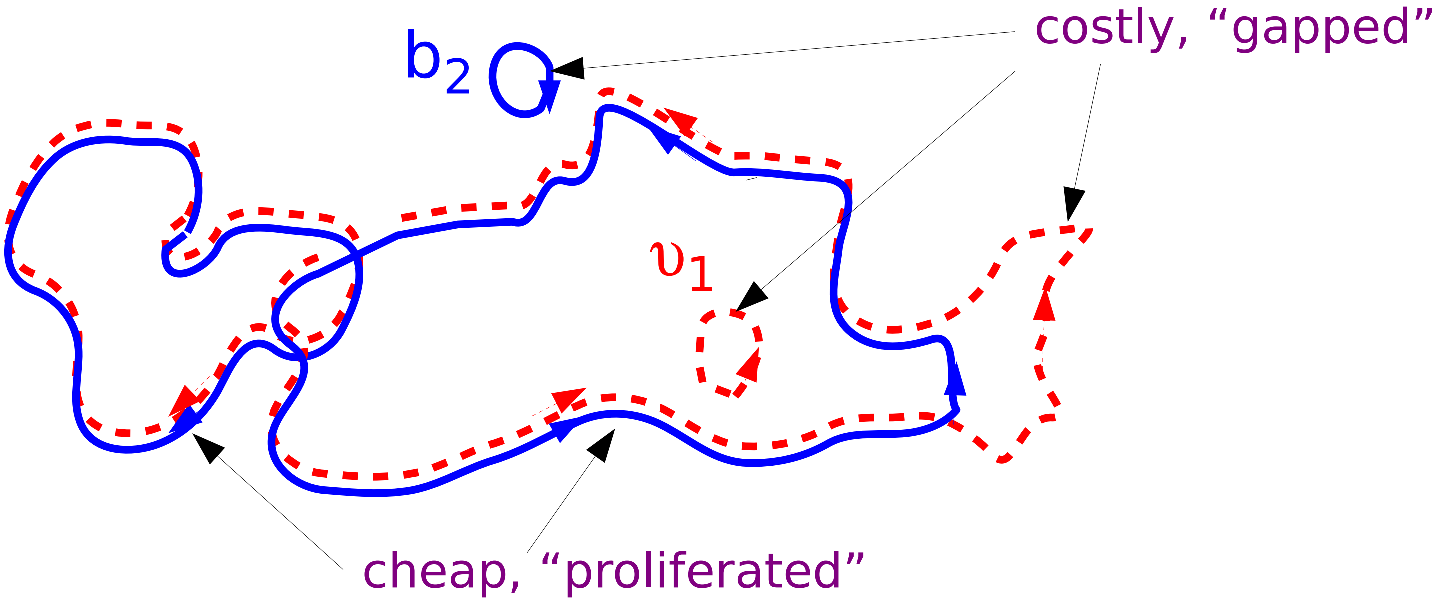
-> excitations with fractional charges (1/d, 0) and (0, 1/d), with mutual statistics $2\pi b/d$, where ad – bc = 1

Engineering condensation of bound states of υ_1 and b_2

$$S[\vec{Q}_1, \vec{J}_2] = \sum_k \frac{1}{2} \frac{(2\pi)^2 \lambda_1}{k^2} |\vec{Q}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2} |\vec{Q}_1 - \vec{J}_2|^2$$

small λ_2: want $Q_1 = J_2$ ---> "binding" to form $(Q_1, J_2) = (1, 1)$

small λ_1: little additional cost for any "bound" configurations ---> "condensation" of bound states (to be precisely defined)



Engineering condensation of bound states of $(\upsilon_1)^d$ and $(b_2)^c$

$$S[\vec{Q}_1, \vec{J}_2] = \sum_k \frac{1}{2}\frac{(2\pi)^2\lambda_1}{k^2}|\vec{Q}_1(k)|^2 + \sum \frac{1}{2\lambda_2 d^2}|c\vec{Q}_1 - d\vec{J}_2|^2$$

small λ_1 and λ_2 ---> condensation of bound states $(Q_1, J_2) = (d, c)$

<u>Precise definition of such bound state condensation:</u>

* Change of variables to new independent loop variables

$Q_1, J_2 \rightarrow F_1 = a{*}Q_1 - b{*}J_2$,

$G_2 = c{*}Q_1 - d{*}J_2$ -- “modular transformation”-- SL(2,Z) matrix

- ok if (a, b, c, d) are integers satisfying $ad - bc = 1$.

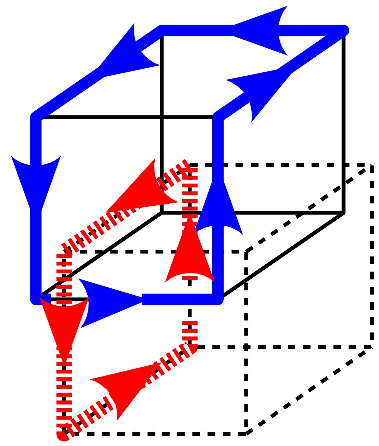
* Expect G_2 gapped and F_1 condensed -> dual G_1 gapped;

duality transform $F_1, G_2 \rightarrow G_1, G_2$ produces action

$$S[\vec{G}_1, \vec{G}_2] = \sum \left[\frac{1}{2\lambda_1 d^2}\vec{G}_1^2 + \frac{1}{2\lambda_2 d^2}\vec{G}_2^2\right] + i\sum \frac{2\pi b}{d}\vec{G}_1 \cdot \vec{a}_{G2}$$

G_1, G_2 describe gapped excitations with mutual statistics $2\pi b/d$

Reverse-engineering physical models



* Conserved integer-valued currents residing on links of inter-penetrating 3D cubic lattices dual to each other

* Euclidean action with local interactions:

$$S = \frac{1}{2}\sum_{r,r'} v_1(r-r')\vec{J}_1(r)\cdot\vec{J}_1(r') + \frac{1}{2}\sum_{R,R'} v_2(R-R')\vec{J}_2(R)\cdot\vec{J}_2(R')$$
$$+i\sum_{R,R'} w(R-R')[\vec{\nabla}\times\vec{J}_1](R)\cdot\vec{J}_2(R')$$

Is this action unitary?
YES – obtainable from a local Hamiltonian!

* Specific short-ranged "potentials" (reverse-engineered):

$$v_{1/2}(k) = \frac{\lambda_{2/1}}{\lambda_1\lambda_2 + \frac{c^2|\vec{k}|^2}{d^2(2\pi)^2}}, \qquad w(k) = \frac{-c}{2\pi d}\frac{1}{\lambda_1\lambda_2 + \frac{c^2|\vec{k}|^2}{d^2(2\pi)^2}}$$

- upon duality on one species J_1, J_2 -> Q_1, J_2 reproduce precisely the "binding" action $S[Q_1, J_2]$

Solution of the physical model; properties

$$S[\vec{J}_1, \vec{J}_2] = S_{\text{v1-v2-w}}[\vec{J}_1, \vec{J}_2] + i\sum \vec{J}_1 \cdot \vec{A}_1^{\text{ext}} + i\sum \vec{J}_2 \cdot \vec{A}_2^{\text{ext}}$$

Sequence of transformations:

1) duality on one species: $J_1, J_2 \rightarrow Q_1, J_2$

2) change of variables: $Q_1, J_2 \rightarrow F_1 = a*Q_1 - b*J_2$,

$G_2 = c*Q_1 - d*J_2$;

- ok if (a, b, c, d) are integers satisfying ad – bc =1 (modular matrix)

3) duality on one species: $F_1, G_2 \rightarrow G_1, G_2$

$$S[\vec{G}_1, \vec{G}_2] = \sum \left[\frac{1}{2\lambda_1 d^2} \vec{G}_1^2 + \frac{1}{2\lambda_2 d^2} \vec{G}_2^2 \right] + i\sum \frac{2\pi b}{d} \vec{G}_1 \cdot \vec{a}_{G2}$$
$$-i\sum \frac{c}{2\pi d}[\vec{\nabla} \times \vec{A}_1^{\text{ext}}] \cdot \vec{A}_2^{\text{ext}} - i\sum \frac{1}{d}\left[\vec{G}_1 \cdot \vec{A}_1^{\text{ext}} + \vec{G}_2 \cdot \vec{A}_2^{\text{ext}}\right]$$

* Small λ_1, λ_2 -> gapped G_1, G_2; can read off quasiparticle charges 1/d and mutual statistics $2\pi b/d$, as well as "background" $\sigma^{12}_{xy} = 2c/d$

* gapped G_2 <--> $G_2 \sim 0$; gapped G_1 <--> condensed F_1

--> condensate of bound states of the type $(Q_1, J_2) = (d, c)$

Sign-free reformulation & Monte Carlo study

* Physical boson action

$$S = \frac{1}{2}\sum_{r,r'} v_1(r-r')\vec{J}_1(r)\cdot\vec{J}_1(r') + \frac{1}{2}\sum_{R,R'} v_2(R-R')\vec{J}_2(R)\cdot\vec{J}_2(R')$$
$$+i\sum_{R,R'} w(R-R')[\vec{\nabla}\times\vec{J}_1](R)\cdot\vec{J}_2(R')$$

-- complex-valued -- sign problem in Monte Carlo!

* Action in Q_1 – J_2 variables

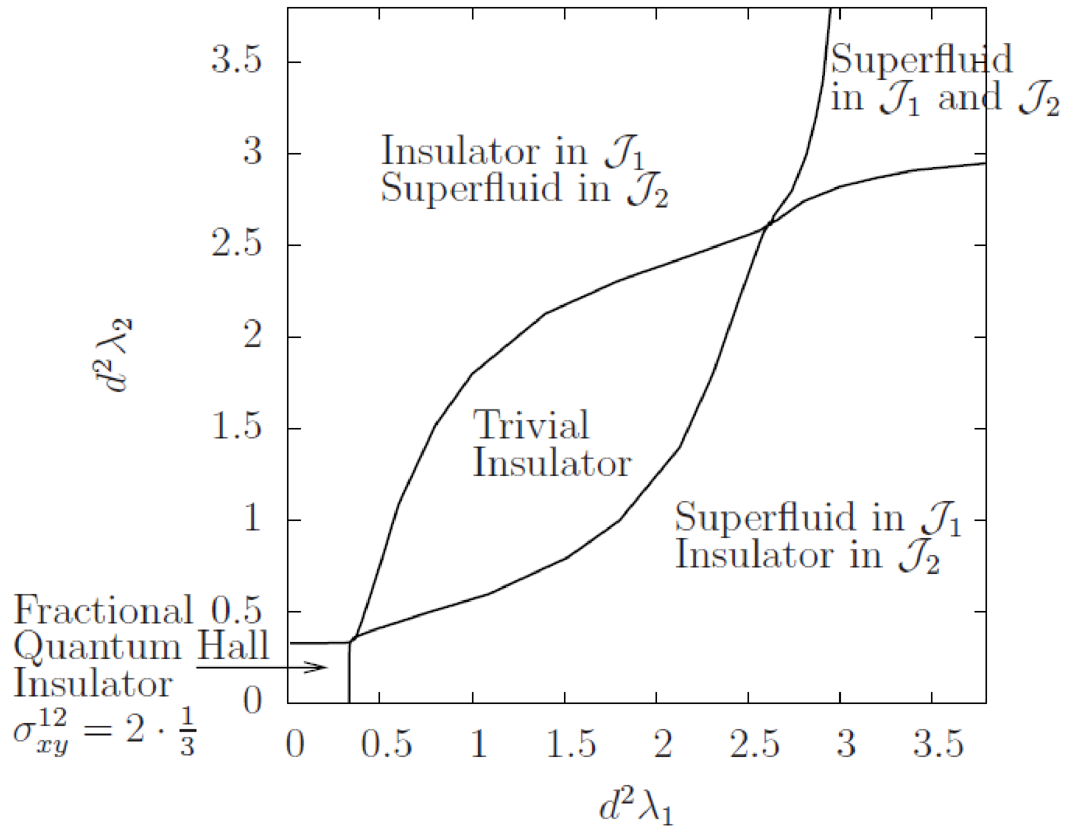
$$S[\vec{Q}_1,\vec{J}_2] = \sum_k \frac{1}{2}\frac{(2\pi)^2\lambda_1}{k^2}|\vec{Q}_1(k)|^2 + \sum \frac{1}{2\lambda_2 d^2}|c\vec{Q}_1 - d\vec{J}_2|^2$$

-- real-valued – can be studied in Monte Carlo!

* Several exact reformulations that can be efficiently simulated
Applications: - phase diagrams and phase transitions
- study of edge states

Monte Carlo study of broader phase diagrams and transitions

Phase diagram for the model with c=1, d=3

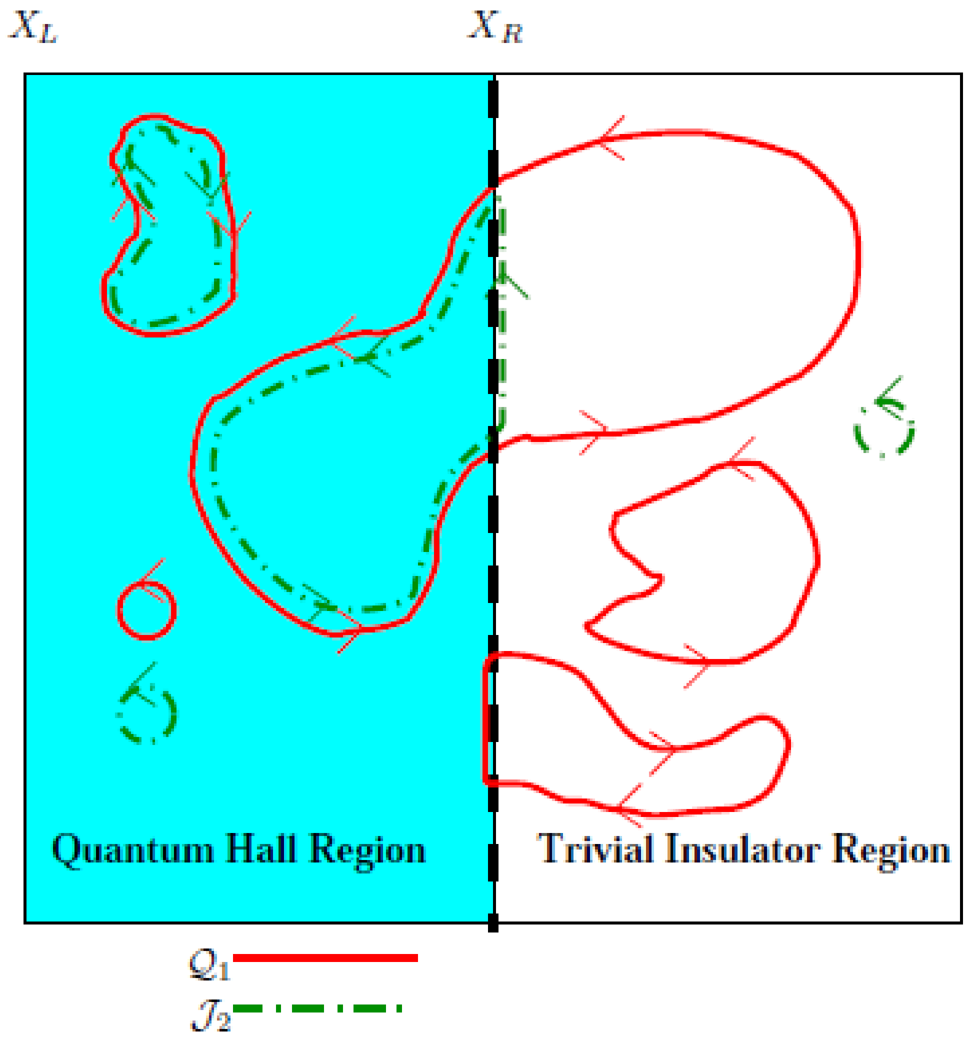


-- can study phase transition between FQH and trivial insulator;
field theory – condensation of quasiparticles with mutual statistics:

$$S = \int_{\mathbb{R}^3} \left[|(\vec{\nabla} - i\vec{\alpha}_1)\Psi_1|^2 + |(\vec{\nabla} - i\vec{\alpha}_2)\Psi_2|^2 + m(|\Psi_1|^2 + |\Psi_2|^2) + \frac{i}{\theta}\vec{\alpha}_1 \cdot (\vec{\nabla} \times \vec{\alpha}_2) \right]$$

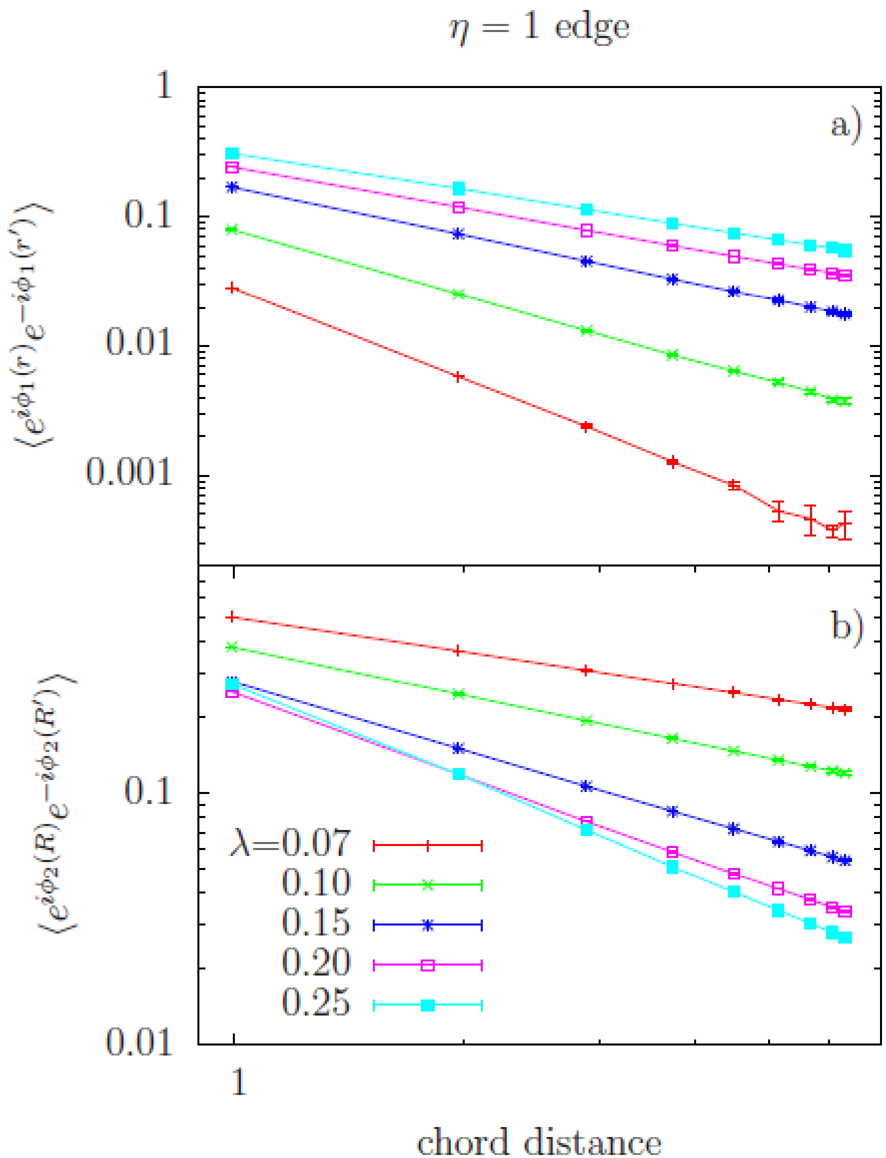
Monte Carlo study of edge states

$$S = \sum_k \frac{1}{2}\frac{(2\pi)^2\lambda_1}{|k|^2}|\vec{Q}_1(k)|^2 + \sum_R \frac{1}{2\lambda_2}|\vec{J}_2(R) - \eta(R)\vec{Q}_1(R)|^2$$

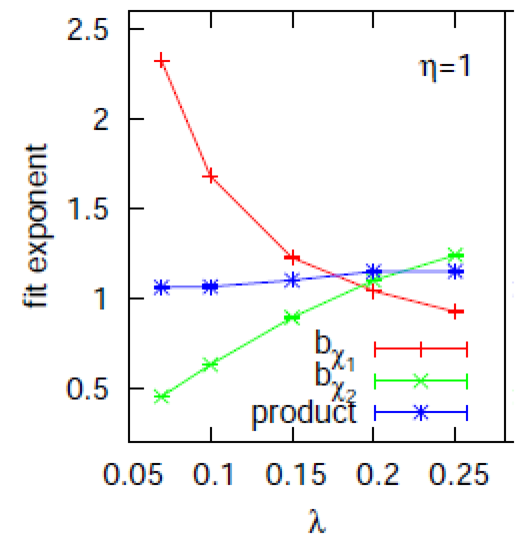


$$\eta(R) = \begin{cases} c/d & \text{for } X_L \le X < X_R \\ 0 & \text{otherwise} \end{cases}$$

Evidence for gapless edge states



-- power-law correlations with oppositely trending exponents for $e^{i\phi 1}$ and $e^{i\phi 2}$



-- consistent with phenomenological edge theory from the K-matrix theory

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow$$

$$S = \int dx d\tau \frac{i}{2\pi} \partial_\tau \varphi_1 \partial_x \varphi_2 + S_{\text{int}}$$

(also studied η=2 and η=1/3 edges)

Highlights and future directions

1) Exact realization of Integer/Fractional Quantum Hall phases of bosons - SPT/SET protected by charge conservation

- Physics: binding of vortices and charges and condensation of composite objects
- Mathematics: generalization of duality to arbitrary modular transformations, giving action in terms of gapped quasi-particles of the phase

2) Sign-free reformulations that can be studied in Monte Carlo

- quantitative phase diagrams of our models
- direct observation of gapless edge states

3) Extensions: other interesting models for SPT/SET phases, similar mechanisms for other symmetries/dimensions:

- Z_NxZ_N in 1d: composites of domain walls and charges
  - SO(3)xU(1) in 3d: composites of hedgehogs and charges (ideas in Vishwanath and Senthil)
- sign-free reformulations of interesting TQFTs?

Robustness & derivation of K-matrix theory

$$S[\vec{J}_1, \vec{J}_2] = S_{\text{short-range}}[\vec{J}_1, \vec{J}_2] + i\sum \vec{J}_1 \cdot \vec{A}_1^{\text{ext}} + i\sum \vec{J}_2 \cdot \vec{A}_2^{\text{ext}}$$

Sequence of transformations:

1) duality on one species: J_1, J_2 -> Q_1, J_2

$$S_1 = S_{s.r.} + i\sum[\frac{\vec{\nabla}\times\vec{\beta}}{2\pi}\cdot\vec{A}_1^{\text{ext}} + \vec{Q}_1\cdot\vec{\beta} + \vec{J}_2\cdot\vec{A}_2^{\text{ext}}]$$

2) change of vars: Q_1, J_2 -> F_1 = a*Q_1 - b*J_2, <-> Q_1 = d*F_1 - b*G_2

G_2 = c*Q_1 - d*J_2 $\qquad$ J_2 = c*F_1 - a*G_2

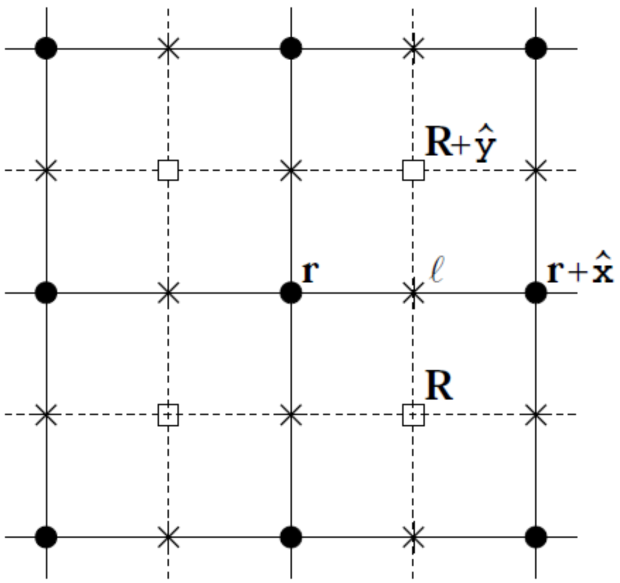
$$S_2 = S_{\text{s.r.}} + i\sum[\frac{\vec{\nabla}\times\vec{\beta}}{2\pi}\cdot\vec{A}_1^{\text{ext}} + \vec{F}_1\cdot(d\vec{\beta} + c\vec{A}_2^{\text{ext}}) - \vec{G}_2\cdot(b\vec{\beta} + a\vec{A}_2^{\text{ext}})]$$

3) duality on one species: F_1, G_2 -> G_1, G_2

$$S_3 = S_{\text{s.r.}} + i\sum[d\frac{\vec{\nabla}\times\vec{\gamma}}{2\pi}\cdot\vec{\beta} + \frac{\vec{\nabla}\times\vec{\beta}}{2\pi}\cdot\vec{A}_1^{\text{ext}} + c\frac{\vec{\nabla}\times\vec{\gamma}}{2\pi}\cdot\vec{A}_2^{\text{ext}}]$$
$$+ i\sum[\vec{G}_1\cdot\vec{\gamma} - \vec{G}_2\cdot(b\vec{\beta} + a\vec{A}_2^{\text{ext}})]$$

- K-matrix-like theory with $K = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ - all properties follow!

Hamiltonian formulation



* conserved bosons of type-1 residing on one square lattice and of type-2 on the dual lattice
* harmonic oscillators {X, P} on the direct and dual link crossings, with X acting as “gauge fields” for the type-1 bosons and P as “gauge fields” for the type-2 bosons:

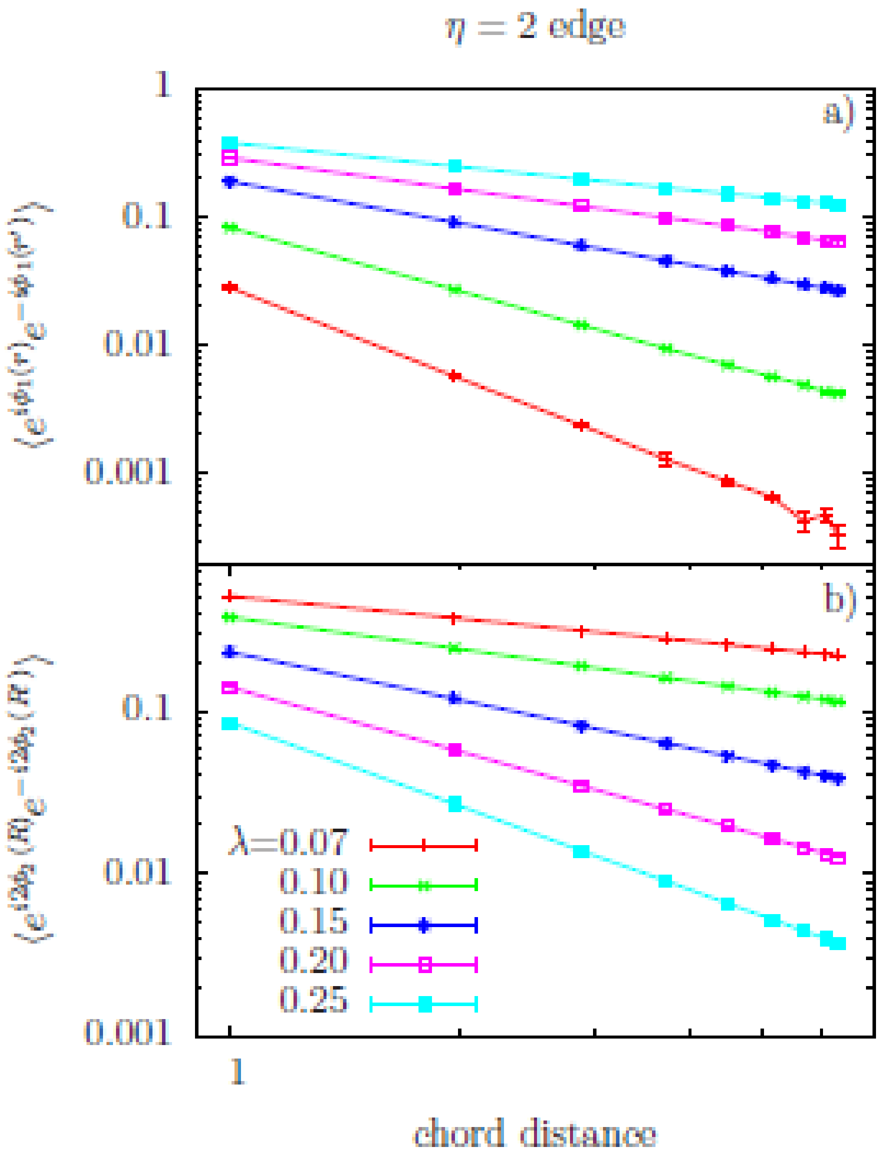
$$\hat{\alpha}_{1\mu} = \hat{\chi}_\mu, \quad \hat{\alpha}_{2\mu} = \epsilon_{\mu\nu}\hat{\pi}_\nu$$

$$H_{\rm bos+osc} = -\sum_{\mathbf{r},j} t\cos[\nabla_j\hat{\phi}_1(\mathbf{r}) - e\hat{\alpha}_{1j}(\mathbf{r})] - \sum_{\mathbf{R},j} t\cos[\nabla_j\hat{\phi}_2(\mathbf{R}) - e\hat{\alpha}_{2j}(\mathbf{R})]$$
$$+\sum_{\mathbf{r}} u\left[\hat{n}_1(\mathbf{r}) + g(\boldsymbol{\nabla}\wedge\hat{\boldsymbol{\alpha}}_2)(\mathbf{r})\right]^2 + \sum_{\mathbf{R}} u\left[\hat{n}_2(\mathbf{R}) + g(\boldsymbol{\nabla}\wedge\hat{\boldsymbol{\alpha}}_1)(\mathbf{R})\right]^2 + \sum_{\ell}\left[\frac{\kappa\,\hat{\chi}_\ell^2}{2} + \frac{\hat{\pi}_\ell^2}{2m}\right]$$

$e/g = 2\pi\, d/c$ -> Euclidean path integral gives precisely our (2+1)d loop models!
Physics: oscillators adjust so that a type-1 boson induces a flux of strength $2\pi\, d/c$ seen by the type-2 bosons. Abrikosov-Nielsen vortex physics:
c type-1 bosons bind d type-2 vortices --- "dynamical flux attachment”

Monte Carlo study of edge states

$$S = \frac{1}{2}\sum_k \frac{(2\pi)^2\lambda_1}{|\vec{f}_k|^2}|\vec{\mathcal{Q}}_1(k)|^2 + \frac{1}{2}\sum_R \frac{1}{\lambda_2}|\vec{\mathcal{J}}_2(R) - \eta(R)\vec{\mathcal{Q}}_1(R)|^2$$

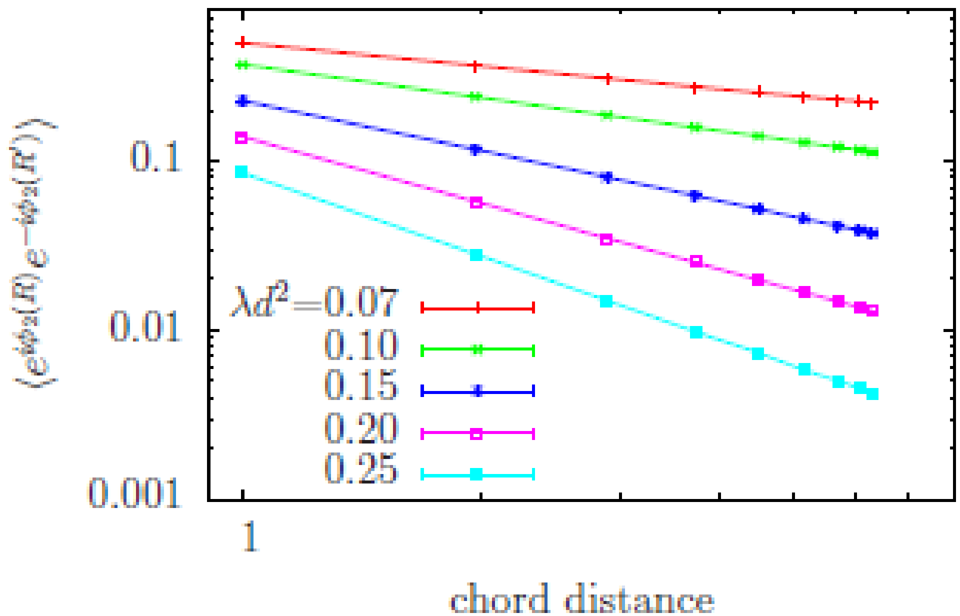


for this specific edge, find power-law correlations for b_1 and "paired" $(b_2)^2$, while b_2 has only exponentially decaying correlations
(thus,can have two distinct "edge phases")

Monte Carlo study of edge states

$$S = \frac{1}{2}\sum_k \frac{(2\pi)^2\lambda_1}{|\vec{f}_k|^2}|\vec{\mathcal{Q}}_1(k)|^2 + \frac{1}{2}\sum_R \frac{1}{\lambda_2}|\vec{\mathcal{J}}_2(R) - \eta(R)\vec{\mathcal{Q}}_1(R)|^2$$

$\eta = \frac{1}{3}$ edge



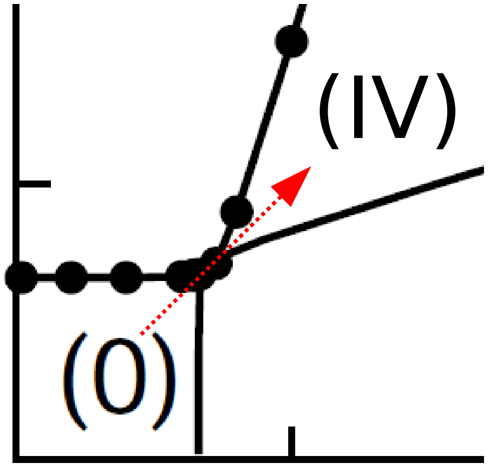
$$K = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \implies S = \int dx d\tau \frac{id}{2\pi}\partial_\tau\varphi_1\partial_x\varphi_2 + S_{\text{int}}$$

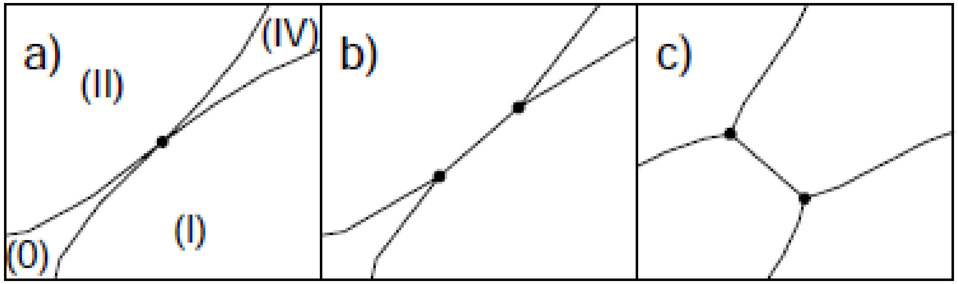
$$b_a \sim e^{id\varphi_a}, \qquad \Delta[b_1]\Delta[b_2] = d^2/4$$

Field theories for the phases and transitions

* Found appropriate gapped variables for each phase -> “long-wavelength field theory” for each phase and also for transitions to proximate phases
* Most transitions are 3D XY in appropriate variables, except the multi-critical points

* Transition (0) -> (IV):





numerics is more consistent with a direct (0)-(IV) transition

If continuous, the field theory for this transition is:

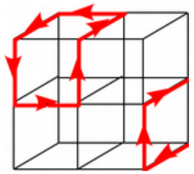
$$S = \int_{\mathbb{R}^3} \left[|(\vec{\nabla} - i\vec{\alpha}_{q1})\Psi_{J1}|^2 + |(\vec{\nabla} - i\vec{\alpha}_{q2})\Psi_{J2}|^2 + m(|\Psi_{J1}|^2 + |\Psi_{J2}|^2) \right]$$

$$+ \int_{\mathbb{R}^3} \left[(\vec{\nabla} \times \vec{\alpha}_{q1})^2 + (\vec{\nabla} \times \vec{\alpha}_{q2})^2 - \frac{i}{\theta} \vec{\alpha}_{q1} \cdot (\vec{\nabla} \times \vec{\alpha}_{q2}) \right]$$

(N.B.: can drop Maxwell terms)

Precise "duality" transform on a 3D lattice

$$Z = \sum_{\vec{J}, \vec{\nabla}\cdot\vec{J}=0} e^{-S[\vec{J}]} = \sum_{\vec{Q}, \vec{\nabla}\cdot\vec{Q}=0} \int D\vec{a}\; e^{-S[\frac{\vec{\nabla}\times\vec{a}}{2\pi}] - i\sum \vec{Q}\cdot\vec{a}}$$



conserved integer-valued 3-currents of bosons on a direct cubic lattice

conserved integer-valued 3-currents of vortices on a dual cubic lattice

"hydrodynamic" real-valued representation of boson 3-current

$$Z = \sum_{\vec{J}, \vec{\nabla}\cdot\vec{J}=0} e^{-\frac{1}{2}\sum_k v(k)|\vec{J}(k)|^2} = \sum_{\vec{Q}, \vec{\nabla}\cdot\vec{Q}=0} e^{-\frac{1}{2}\sum_k \frac{(2\pi)^2}{k^2 v(k)}|\vec{Q}(k)|^2}$$

Schematic continuum theories:

$$\mathcal{L}_{\rm bos} = |\vec{\nabla}\Psi|^2 + m|\Psi|^2 + u|\Psi|^4$$ --- ϕ^4 field theory

$$\mathcal{L}_{\rm vort} = |(\vec{\nabla} - i\vec{a})\Psi_v|^2 + m_v|\Psi_v|^2 + u_v|\Psi_v|^4 + \kappa(\vec{\nabla}\times\vec{a})^2$$ --- Higgs model